A one-parameter representation of credit risk and transition matrices

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This paper presents a one-parameter representation of credit risk and transition matrices. We start with the CreditMetrics view that ratings transition matrices result from the "binning" of a standard normal random variable X that measures changes in creditworthiness. We further assume that X splits into two parts: (1) an idiosyncratic component Y, unique to a borrower, and (2) a systematic component Z, shared by all borrowers. Broadly speaking, Z measures the "credit cycle," meaning the values of default rates and of end-of-period risk ratings not predicted, using historical average transition rates, by the initial mix of credit grades. In good years Z will be positive, implying for each initial credit rating, a lower than average default rate and a higher than average ratio of upgrades to downgrades. In bad years, the reverse will be true. We describe a way of estimating Z from the separate transition matrices tabulated each year by Standard & Poor's (S&P) and Moody's. Conversely, we describe a method of calculating transition matrices conditional on an assumed value for Z.

The historical pattern of Z depicts past credit conditions. For example, Z remains negative for most of 1981–89. This mirrors the general decline in credit ratings over that period. In 1990–91, Z drops well below zero as the U.S. suffers through one of its worst credit crises since the Great Depression. The relatively high proportion of lower grade credits inherited from the 1980s together with the 1990–91 slump ($Z < 0$) accounts for the high number of defaults. Over 1992–97, Z has stayed positive and credit conditions have remained benign. The movements of Z over the past 10 years correlate closely with loan pricing.

Our focus is on how Z affects credit rating migration probabilities. However, one can also model the effect of Z on the probability distribution of loss in the event of default (LIED), on credit par spreads, and ultimately on the value of a commercial loan, bond, or other instrument subject to credit risk. By parametrically varying Z, one can perform stress testing to assess the impact of changing credit conditions on the value of an individual credit instrument or an entire credit portfolio to changing credit conditions. One can also quantify how volatility in Z translates into transaction and portfolio value volatility.

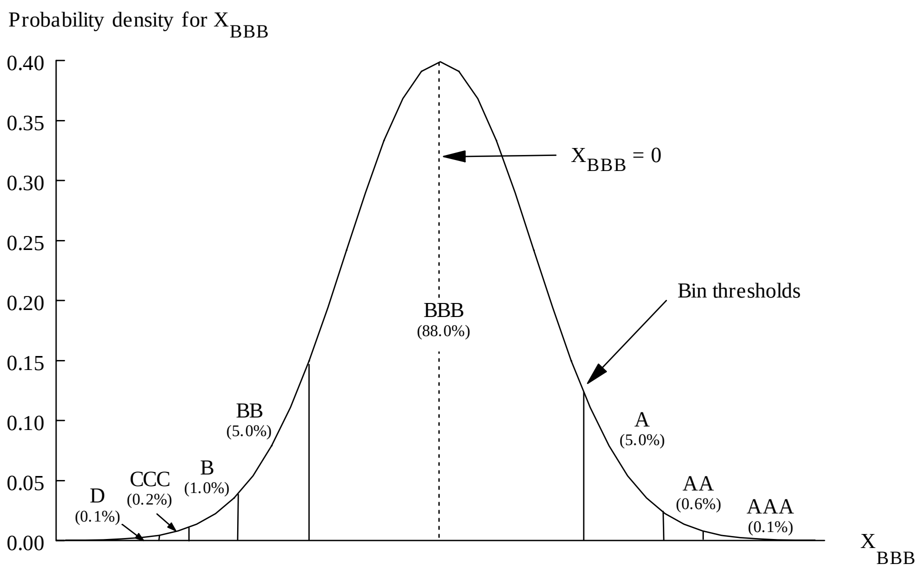
1. Defining *Z* risk

Following the CreditMetrics approach described by Gupton, Finger, and Bhatia (1997), we assume that ratings transitions reflect an underlying, continuous credit-change indicator X. We further assume that X has a standard normal distribution. Then, conditional on an initial credit rating G at the beginning of a year, we partition the X values into a set of disjoint bins $(x_g^G, x_{g+1}^G]$.[1] To simplify references, we use the indices G and g to represent sequences of integers rather than letters or other symbols. We then define the bins such that the probability of X falling within a given interval equals the corresponding historical average transition rate (see Chart 1).

[1] We observe an inconsistency among the bins for different initial ratings. For an initial borrower rating of G_0, consider successive yearly values for X of x_1 and x_2, in which x_1 implies a rating change to G_1 and x_2 a change to G_2. We will not find, in general, that an X value of x_1+x_2 in the first year implies a rating change from G_0 to G_2.

A one-parameter representation of credit risk and transition matrices *(continued)*

Chart 1
Relationship between continuous credit index *X* and rating transitions
Historical average transition rates determine bin thresholds



We write the conditions defining the bins as follows:

[1] $$P(G, g) = \Phi(x^G_{g+1}) - \Phi(x^G_g)$$

in which $P(G, g)$ denotes the historical average G-to-g transition probability and $\Phi(\cdot)$ represents the standard normal cumulative distribution function. The default bin D has a lower threshold of $-\infty$. The AAA bin has an upper threshold of $+\infty$. The remaining thresholds are fit to the observed transition probabilities.

Suppose there are N ratings categories, including default. Then there are $N-1$ initial grades, which represent all the ratings, excluding default. For each of those initial grades, we observe $N-1$ historical average transition rates. The Nth value results from the condition that the probabilities sum to 1. We must determine $N-1$ threshold values defining the bins. Thus, we can solve for all of the bin boundaries.

We illustrate the process below. The starting point is the smoothed version of the 1981–97 historical average transition matrix tabulated by S&P for 8 grades, including default (see Table 1). The corresponding bins are computed using Eq. [1].[2]

[2] The smoothing applied to the matrix enforces default rate monotonicity, row and column monotonicity and several of the other regularity conditions listed in the *CreditMetrics™—Technical Document*. Default rate monotonicity means that default rates rise as credit ratings go down. Row and column monotonicity means that transition rates fall as one moves away from the main diagonal along either a row or a column. We note one exception to this rule. Default is a trapping state. Thus, the default rate may rise above the probability of transition to neighboring non-default states.

A one-parameter representation of credit risk and transition matrices *(continued)*

Consider transitions from BBB. We observe a 15 bp default rate. Using the inverse probability function for a standard normal distribution, we compute a value of about −2.97 for the upper threshold for the default bin. Next consider the CCC bin. We get a value of about 25 bp for the *sum* of transition rates to CCC or to default. Again applying the inverse probability function, we get an upper threshold value for CCC of about −2.81. Now consider B. We compute a probability of about 1.3 percent for transitions to B or to lower grades. Once again applying the inverse probability function, we get an upper threshold value of −2.23. Continuing in this way for each terminal and each initial grade, we derive all of the bin values.

Table 1
Smoothed historical average transition matrix and associated bins

	Initial rating	**End-of-year credit rating**							
		AAA	AA	A	BBB	BB	B	CCC	D
Smoothed historical average transition matrix	AAA	91.13%	8.00%	0.70%	0.10%	0.05%	0.01%	0.01%	0.01%
	AA	0.70%	91.03%	7.47%	0.60%	0.10%	0.07%	0.02%	0.01%
	A	0.10%	2.34%	91.54%	5.08%	0.61%	0.26%	0.01%	0.05%
	BBB	0.02%	0.30%	5.65%	87.98%	4.75%	1.05%	0.10%	0.15%
	BB	0.01%	0.11%	0.55%	7.77%	81.77%	7.95%	0.85%	1.00%
	B	0.00%	0.05%	0.25%	0.45%	7.00%	83.50%	3.75%	5.00%
	CCC	0.00%	0.01%	0.10%	0.30%	2.59%	12.00%	65.00%	20.00%
Bins corresponding to smoothed historical average transition matrix	AAA	(∞, −1.35)	[−1.35, −2.38)	[−2.38, −2.93)	[−2.93, −3.19)	[−3.19, −3.54)	[−3.54, −3.72]	[−3.72, −3.89)	[−3.89, −∞)
	AA	(∞, 2.46)	[2.46, −1.39)	[−1.39, −2.41)	[−2.41, −2.88)	[−2.88, −3.09)	[−3.09, −3.43)	[−3.43, −3.72)	[−3.72, −∞)
	A	(∞, 3.10)	[3.10, 1.97)	[1.97, −1.55)	[−1.55, −2.35)	[−2.35, −2.73)	[−2.73, −3.24)	[−3.24, −3.29)	[−3.29, −∞)
	BBB	(∞, 3.50)	[3.50, 2.73)	[2.73, 1.56)	[1.56, −1.55)	[−1.55, −2.23)	[−2.23, −2.81)	[−2.81, −2.97)	[−2.97, −∞)
	BB	(∞, 3.89)	[3.89, 3.05)	[3.05, 2.48)	[2.48, 1.38)	[1.38, −1.29)	[−1.29, −2.09)	[−2.09, −2.33)	[−2.33, −∞)
	B	(∞, 4.11)	[4.11, 3.29)	[3.29, 2.75)	[2.75, 2.43)	[2.43, 1.42)	[1.42, −1.36)	[−1.36, −1.64)	[−1.64, −∞)
	CCC	(∞, 4.27)	[4.27, 3.72)	[3.72, 3.06)	[3.06, 2.64)	[2.64, 1.88)	[1.88, 1.04)	[1.04, −0.84)	[−0.84, −∞)

As in Belkin, Suchower, and Forest (1998), we decompose X into two parts: (1) a (scaled) idiosyncratic component Y, unique to a borrower, and (2) a (scaled) systematic component Z, shared by all borrowers. Thus, we write

[2] $$X = \sqrt{1-\rho}\,Y + \sqrt{\rho}\,Z$$

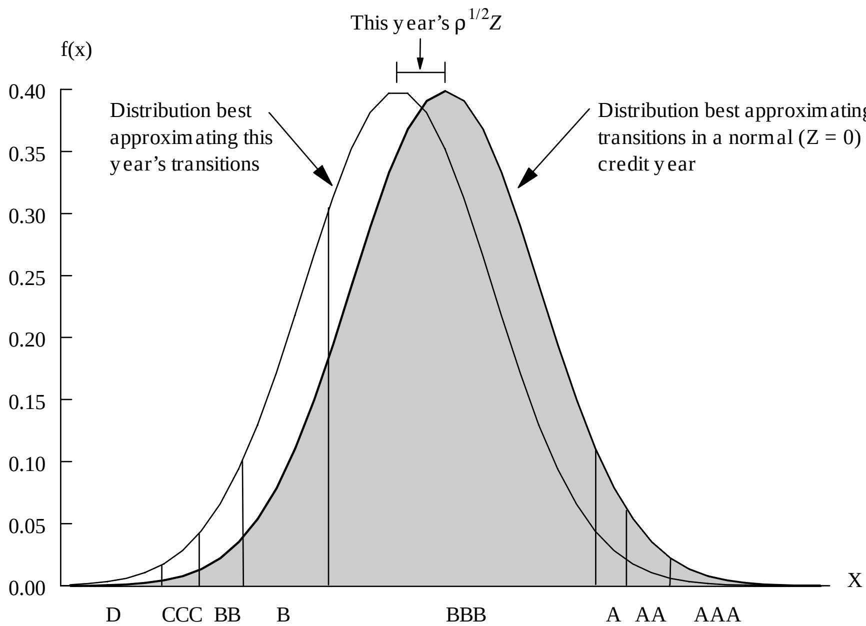
We assume that Y and Z are unit normal random variables and mutually independent.[3] The parameter ρ (assumed positive) drives the correlation between Z and X; Z explains a fraction ρ of the variance of X.

In any year, the observed transition rates will deviate from the norm ($Z = 0$). We can then find a value of Z such that the probabilities associated with the bins defined above best approximate the given year's observed transition rates (see Chart 2).

[3] The variate Z actually changes from year to year, and is modeled as following a stochastic process; therefore, it is more proper to denote it by Z_t. A reasonable stochastic model is the Ornstein-Uhlenbeck (O-U) process $dZ_t = -\beta Z_t dt + \sigma dW_t$ with parameters β (reciprocal of time constant) and σ (volatility); W_t is a standard Wiener process. The O-U process is mean reverting (capturing the analogous property of the business cycle) and has a limiting stationary Gaussian distribution. The condition $\sigma^2/2\beta = 1$ is imposed to insure that the stationary distribution has unit variance. See Arnold (1974) for a discussion of the O-U process.

A one-parameter representation of credit risk and transition matrices *(continued)*

Chart 2
Illustration of the *Z* value for a particular initial rating in a given year



We label that value of Z for year t, Z_t.[4] We determine Z_t so as to minimize the weighted, mean-squared discrepancies between the model transition probabilities and the observed transition probabilities.

For this we define

[3] $$\Delta(x^G_{g+1}, x^G_g, Z_t) = \Phi\left(\frac{x^G_{g+1} - \sqrt{\rho}Z_t}{\sqrt{1-\rho}}\right) - \Phi\left(\frac{x^G_g - \sqrt{\rho}Z_t}{\sqrt{1-\rho}}\right) .$$

This is the fitted value for the G-to-g transition rate in year t. Then for a fixed ρ and a fixed t, the least-squares problem takes the form

[4] $$\min_{Z_t} \sum_G \sum_g \frac{n_{t,g}[P_t(G,g) - \Delta(x^G_{g+1}, x^G_g, Z_t)]^2}{\Delta(x^G_{g+1}, x^G_g, Z_t)[1 - \Delta(x^G_{g+1}, x^G_g, Z_t)]} ,$$

4 It can be shown that one recovers the historical average transition matrix by integrating the transition matrices conditioned on $Z_t = z$ with respect to the stationary unit normal distribution for z. Thus, the historical average matrix is the expectation of the conditioned matrices over all possible values of Z.

A one-parameter representation of credit risk and transition matrices *(continued)*

where $P_t(G, g)$ represents the G-to-g transition rate observed in year t and $n_{t,G}$ is the number of transitions from initial grade G observed in that year. In this formula, we weight observations by the inverses of the approximate sample variances of $P_t(G, g)$.[5]

Since we do not know the value of ρ a priori, we estimate it as follows. We apply the minimization in Eq. [4] for 1981–97 using an assumed value of ρ. We then obtain a time series for Z_t conditional on ρ and compute the mean and variance of this series. We repeat this process for many values of ρ, and use a numerical search procedure to find the particular ρ value for which the Z_t time series has variance of one.

We illustrate this process of solving for Z_t at a single time t. We start with the S&P transition matrix observed for 1982 (see Table 2). We hold fixed the bins determined from the historical average matrix (Table 1) and fix ρ at the value determined by the search process, i.e., .0163. The indicated value for Z_t of –0.89 provides the best fit to the observed 1982 transition rates.

Table 2
S&P transition matrix for 1982 and calculations leading to *Z* estimate

Statistic	**Initial rating**	**# Obs.**	**End-of-year credit rating** AAA	AA	A	BBB	BB	B	CCC	D
Observed transition matrix	AAA	85	92.94%	4.71%	2.35%	0.00%	0.00%	0.00%	0.00%	0.00%
	AA	220	0.46%	92.52%	6.08%	0.47%	0.47%	0.00%	0.00%	0.00%
	A	480	0.00%	4.45%	84.95%	9.54%	0.64%	0.00%	0.00%	0.42%
	BBB	298	0.37%	0.37%	3.26%	85.52%	9.78%	0.37%	0.00%	0.34%
	BB	168	0.00%	0.68%	0.00%	2.68%	82.42%	10.05%	0.00%	4.17%
	B	161	0.00%	0.00%	0.72%	0.72%	2.89%	87.50%	5.06%	3.11%
	CCC	16	0.00%	0.00%	0.00%	0.00%	0.00%	7.39%	73.86%	18.75%
Fitted transition matrix	AAA	85	89.34%	9.54%	0.89%	0.13%	0.07%	0.01%	0.01%	0.01%
	AA	220	0.48%	89.56%	8.93%	0.77%	0.13%	0.09%	0.03%	0.01%
	A	480	0.06%	1.72%	90.88%	6.14%	0.78%	0.34%	0.01%	0.07%
	BBB	298	0.01%	0.20%	4.39%	88.03%	5.72%	1.33%	0.13%	0.20%
	BB	168	0.00%	0.07%	0.38%	6.19%	81.63%	9.39%	1.06%	1.29%
	B	161	0.00%	0.03%	0.17%	0.32%	5.56%	83.41%	4.38%	6.14%
	CCC	16	0.00%	0.01%	0.06%	0.20%	1.94%	10.09%	64.53%	23.16%
Z value	−0.89									

Broadly speaking, Z_t measures the "credit cycle," meaning the values of default rates and of end-of-period risk ratings not predicted (using historical average transition rates) by the initial mix of credit grades. In good years Z_t will be positive, implying for each initial credit rating a lower than average default rate and a higher than average ratio of upgrades to downgrades. In bad years, the reverse will be true.

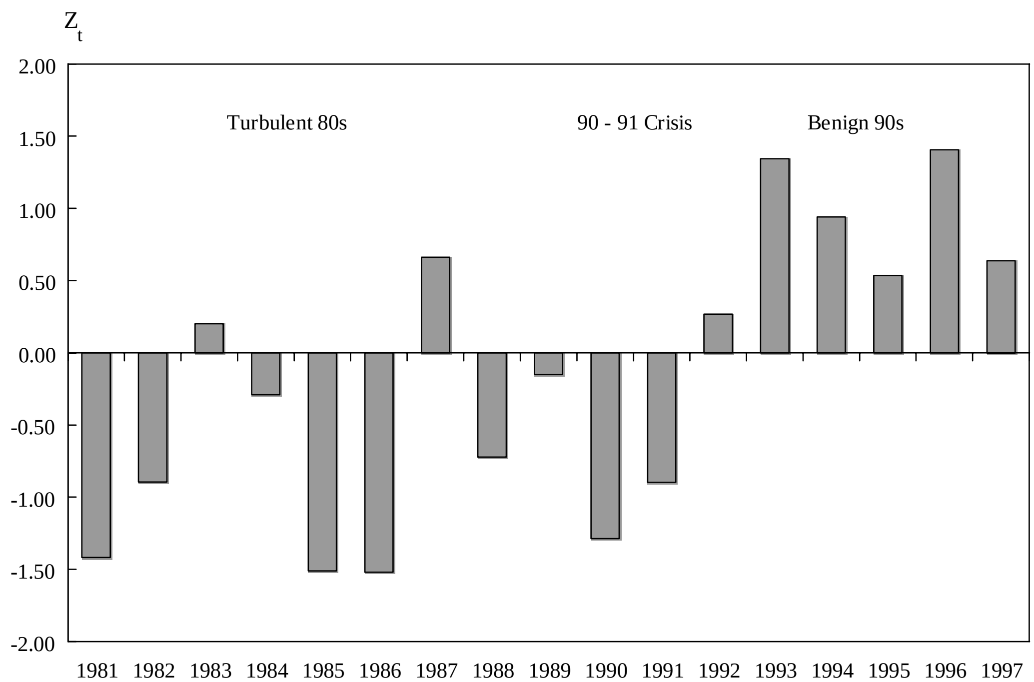
5 In Eq. [4], we normalize each squared deviation by the factor $\frac{\Delta(x^G_{g+1}, x^G_g, Z_t)[1-\Delta(x^G_{g+1}, x^G_g, Z_t)]}{n_{t,G}}$. This weighting factor represents the sample variance for the G-to-g transition rate under a binomial sampling approximation such that "success" is the occurrence of a G-to-g transition and "failure" is any other transition. A full multinomial treatment would account for the constraint that the sample transition rates across a row must sum to one.

A one-parameter representation of credit risk and transition matrices *(continued)*

2. Z_t's historical patterns

Z_t's historical movements describe past credit conditions not evident in the initial mix of ratings (see Chart 3). Z_t's history is erratic, more so than the term "credit cycle" suggests. In particular, the fluctuations do not show a stable sinusoidal pattern.

Chart 3
Z_t as estimated from S&P annual transition matrices

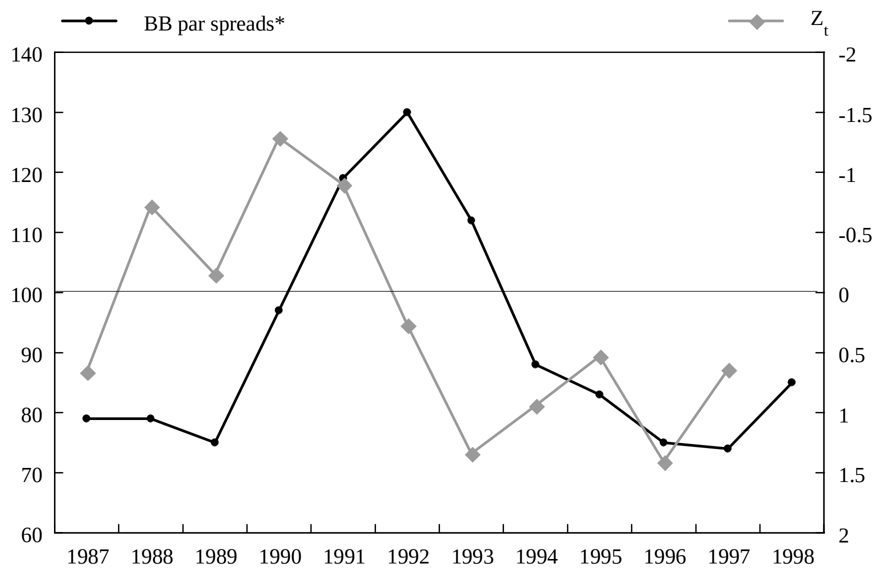


Z_t is mostly negative over 1981–89. Credit ratings generally declined over that period as many corporations increased leverage. In 1990–91, Z_t drops below zero, as the U.S. suffers through one of its worst credit crises since the Great Depression. The relatively high proportion of lower grade credits inherited from the 1980s together with the 1990–91 credit slump ($Z_t < 0$) accounts for a high number of defaults. Over the period 1992–97, Z_t has stayed positive and credit conditions have remained benign.

Loan prices over the past 10 years correlate quite closely with the credit indicator Z_t (see Chart 4). One observes that loan spreads have generally lagged abrupt changes in credit conditions.

A one-parameter representation of credit risk and transition matrices *(continued)*

Chart 4
Z_t index and BB spreads



* *Source*: Loan Pricing Corporation.

Loan spreads in North America and Europe over the past 2–3 years have remained near record lows. This suggests that the past 6 years of favorable credit conditions have made many lenders optimistic about the future. One might ask whether the past patterns exhibited by Z_t justify this optimism or any other forecast of credit conditions.

Applying the weighted least-squares scheme for estimating the Z_t, we get a ρ value of 0.0163. Thus, systematic credit migration risk accounts for only about 1.6% of total credit migration risk over the period 1981–97. This contrasts with equity price data, which suggests that systematic risk accounts for about 25 percent of the total variance in an average company's stock price. Still, the seemingly small estimated variations in Z_t translate into substantial swings in default and downgrade rates (see the discussion following Table 3 on page 54).

The discrete time counterpart to the O-U model is a first-order autoregressive process. We fitted such a model for Z_t to the data for the 1982–97 period and obtained the following:

[5] $$Z_t - Z_{t-1} = -0.54 Z_{t-1} + 1.04\varepsilon_t .$$

Here ε_t is a standardized white noise sequence. The sample mean of the Z_t values is –0.16, which is statistically consistent with the assumption that the Z_t process has zero mean. From Eq. [5] we obtain the sample estimates $\beta = .54\ \text{yr}^{-1}$ and $\sigma = 1.04\ \text{yr}^{-1/2}$ for the O-U process parameters. Thus, the Z_t process has an estimated mean relaxation time of about 2 years and an estimated annual volatility of about 1.

We performed several statistical tests on Eq. [5] for model goodness of fit. The sample estimate of the mean of the Z_t process mean is –0.16, with a standard error of 0.25. As a result, there is no statistical basis to reject the hypothesis that the Z_t process has zero mean. Based on a t-statistic value of 2.18, the hypothesis that there is no mean reversion (i.e., that $\beta = 0$) can be rejected at the .025 significance level. The Kolmogorov-Smirnov test statistic for the model residuals has a value of d = .17, indicating that the residuals are statistically indistinguishable from a white noise sequence ($\alpha = .71$).

The calculated R^2 for the model in Eq. [5] is .24, indicating that a first-order autoregressive model for the Z_t has modest predictive power.[6] However, the utility of the model is not in predicting future values of Z_t. Rather, it is to quantify how the variability in Z_t that is predictable and the variability in Z_t that is not predictable each influence credit risk and the pricing of that risk.

3. Determining transition matrices as functions of Z_t

We have already described a way of imputing the Z_t variable from observed transition matrices. By inverting this process, we can determine transition matrices from values of Z_t.

We again use the bin values x_G for each initial grade *G* and end-of-year grade *g*. Now, conditional on Z_t, we compute the probability of a G-to-g transition as

[6]
$$P_t(G, g) = \Phi\left(\frac{x^G_{g+1} - \sqrt{\rho}Z_t}{\sqrt{1-\rho}}\right) - \Phi\left(\frac{x^G_g - \sqrt{\rho}Z_t}{\sqrt{1-\rho}}\right)$$

Table 3 shows matrices for a good year ($Z_t = 1$), an average year ($Z_t = 0$), and a bad year ($Z_t = -1$). Note that an absolute value of 1 for Z_t represents a 1-standard deviation variation from “normal” credit conditions.

[6] The R^2 for a second-order autoregressive model is only .33, so going to a higher order model adds little in the way of predictive power. The simply reality is that the Z_t process, at least over the 17-year historical period analyzed, is quite volatile.

A one-parameter representation of credit risk and transition matrices *(continued)*

Table 3
Transition matrices computed using *Z* parameterization

Statistic	Initial rating	End-of-year credit rating AAA	AA	A	BBB	BB	B	CCC	D
Calculated transition matrix for good year ($Z_t = 1$)	AAA	93.17%	6.25%	0.47%	0.06%	0.03%	0.01%	0.00%	0.00%
	AA	0.95%	92.72%	5.81%	0.41%	0.06%	0.04%	0.01%	0.01%
	A	0.14%	3.02%	92.33%	3.88%	0.42%	0.17%	0.01%	0.03%
	BBB	0.03%	0.41%	7.03%	88.00%	3.65%	0.73%	0.06%	0.09%
	BB	0.01%	0.15%	0.73%	9.50%	82.00%	6.32%	0.61%	0.67%
	B	0.00%	0.07%	0.34%	0.59%	8.58%	83.68%	3.03%	3.70%
	CCC	0.00%	0.01%	0.14%	0.40%	3.30%	14.12%	65.60%	16.42%
Calculated transition matrix for average year ($Z_t = 0$)	AAA	91.31%	7.87%	0.67%	0.09%	0.05%	0.01%	0.00%	0.00%
	AA	0.66%	91.24%	7.34%	0.57%	0.09%	0.06%	0.02%	0.01%
	A	0.09%	2.26%	91.79%	4.98%	0.58%	0.24%	0.01%	0.05%
	BBB	0.02%	0.28%	5.52%	88.28%	4.66%	1.01%	0.09%	0.14%
	BB	0.00%	0.10%	0.52%	7.63%	82.13%	7.84%	0.82%	0.95%
	B	0.00%	0.04%	0.23%	0.43%	6.87%	83.85%	3.71%	4.86%
	CCC	0.00%	0.01%	0.09%	0.28%	2.51%	11.91%	65.39%	19.81%
Calculated transition matrix for bad year ($Z_t = -1$)	AAA	89.09%	9.75%	0.92%	0.14%	0.07%	0.01%	0.01%	0.01%
	AA	0.46%	89.34%	9.13%	0.79%	0.14%	0.10%	0.03%	0.01%
	A	0.06%	1.66%	90.75%	6.28%	0.80%	0.35%	0.01%	0.07%
	BBB	0.01%	0.19%	4.27%	87.96%	5.85%	1.37%	0.14%	0.21%
	BB	0.00%	0.07%	0.37%	6.03%	81.53%	9.58%	1.09%	1.33%
	B	0.00%	0.03%	0.16%	0.31%	5.42%	83.32%	4.47%	6.30%
	CCC	0.00%	0.00%	0.06%	0.20%	1.88%	9.88%	64.39%	23.58%

One observes here significant variation in the migration probabilities between the good, average, and bad years, particularly off the main diagonal. For example, the default probability starting in grade B is .0486 in an average year but increases to .063 in a bad year and drops to .037 in a good year. In relative terms these are about 30% variations. Consequently, the effect of systematic risk on migration probabilities is significant.

4. Applications

The Z variable and its related formulas provide a simple one-factor description of credit portfolio risk and credit pricing. On the pricing side, changes in credit spreads for a given grade reflect shifting expectations regarding expected and unexpected loss. By “unexpected loss,” we mean the *premium* (over expected loss) that a loan must pay to compensate for its contribution to volatility in a well-diversified portfolio.

We can explain changes in credit spreads using Z. Suppose that the expected value of Z increases. Then the anticipated transition rates to default and to near default go down. The probability distribution for LIED can shift downward (and change shape) as well. The effect is that both expected and unexpected losses fall, lowering credit spreads. Suppose, alternatively, that the expected value of Z decreases. Expected and unexpected losses go up, raising credit spreads. Thus, we can relate spread volatility to changes in expected credit conditions.

Given a stochastic specification for Z, we can incorporate spread volatility into loan pricing models. We are currently modifying KPMG’s Loan Analysis System℠ by incorporating Z risk along with its

A one-parameter representation of credit risk and transition matrices *(continued)*

effect on rating migration probabilities, on the distribution of LIED, and on par credit spreads. In addition, we are including Z risk as one of the factors in a multifactor model for the interest rate term structure.

The Z variable provides a simple way of running credit scenarios. For example, one might want to simulate the value of a credit portfolio under conditions similar to those in 1990–91. To accomplish this, one would run a two-year simulation, setting Z equal to its 1990 value in year 1 and to its 1991 value in year 2. One would compute the associated transition matrices and use those matrices in calculating credit value-at-risk.

Alternatively, one could run a large number of simulations drawing Z from a time series model, such as the O-U process. This would provide valuable insight into how volatility in Z in response to changing credit conditions induces volatility in the mark-to-market value of a credit portfolio.

In closing, we note that Z offers only a one-factor explanation of credit risk. International data suggest that one needs several factors to describe credit risk globally. The assumption that a single factor can satisfactorily represent all systematic risk in valuing a credit portfolio needs to be tested by comparing model predictions of mark-to-market prices with observed market prices.

5. Summary

We have described a one-parameter representation of credit risk and transition matrices in the form of a single systematic credit factor Z. The historical record of Z provides a succinct description of past credit conditions. We have described a stochastic process model for Z and a way of estimating Z from past ratings transition matrices and applied the method to rating migration data for the period 1983–97.

Our results indicate that specific risk dominates systematic risk in terms of explaining the variance of X, the continuous variate that governs credit migration under the CreditMetrics model. Nonetheless, Z has a significant effect on migration probabilities, and the framework that we described can be used to stress test a credit portfolio, i.e., to quantify the impact of changing credit conditions on individual transaction value and portfolio value.

The Z variate can be incorporated into models for stochastic LIED and stochastic par credit spreads. It also provides a basis for modeling the correlation between credit migration and interest rates, foreign currency exchange rates, and other market variables subject to systematic risk.

A one-parameter representation of credit risk and transition matrices *(continued)*

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